AIAA JOURNAL
Vol. 40, No. 6, June 2002

Enforcing Structural Connectivity to Update Damped Systems Using Frequency Response

Philip D. Cha* and Joshua P. Switkes†
Harvey Mudd College, Claremont, California 91711

To benefit computational model validation, improve active vibration control algorithms, and aid damage detection for aging structural systems, new approaches are developed to update the analytical system matrices of a damped structure using frequency response. When the difference between the measured frequency response data and the analytical predictions is used and the resulting matrix equations are manipulated, the mass, stiffness, and damping correction matrices can be isolated in turn. When these correction matrices are rearranged into vector forms, the readily available structural connectivity information of the analytical matrices can be enforced, thereby preserving the physical configuration of the system and reducing the size of the least-squares problems that need to be solved. The required solution techniques to perform the model update are introduced, and the numerical issues associated with solving overdetermined and underdetermined least-squares problems are investigated. Heuristic criteria are given for determining the minimum number of frequency-response data points that need to be measured to ensure sufficiently accurate updated system matrices, and numerical experiments are performed to validate the proposed model updating techniques based on using the frequency-response data.

Nomenclature

$[C]$	=	actual damping matrix
$[C_{\text{update}}]$	=	updated damping matrix
$[C_0]$	=	analytical damping matrix
$\boldsymbol{f}$	=	vector of forcing amplitudes
j	=	imaginary unit
$[K]$	=	actual stiffness matrix
$[K_{\text{update}}]$	=	updated stiffness matrix
$[K_0]$	=	analytical stiffness matrix
$[M]$	=	actual mass matrix
$[M_{\text{update}}]$	=	updated mass matrix
$[M_0]$	=	analytical mass matrix
$\boldsymbol{m}_{\text{actual}}$	=	vector of actual masses
$\boldsymbol{m}_{\text{analytical}}$	=	vector of analytical masses
$\boldsymbol{m}_{\text{update}}$	=	vector of updated masses
N	=	degrees of freedom of the analytical model
n	=	number of distinct frequency response data points
$[X]$	=	$N \times n$ matrix whose ith column corresponds to $\bar{\boldsymbol{x}}$ at ω_i
$[X_{\mathcal{I}}]$, $[X_{\mathcal{R}}]$	=	imaginary and real parts of $[X]$
$\boldsymbol{x}$	=	measured output vector of the actual system
$\boldsymbol{x}_0$	=	output vector of the analytical system
$\bar{\boldsymbol{x}}$	=	complex steady state amplitude of $\boldsymbol{x}$
$[\delta C]$	=	damping correction matrix
$\delta\boldsymbol{c}$	=	damping correction vector
δc_i	=	percent deviation of the ith actual damping parameter from its analytical value
$\delta\boldsymbol{c}'$	=	reduced damping correction vector
$[\delta K]$	=	stiffness correction matrix
$\delta\boldsymbol{k}$	=	stiffness correction vector
δk_i	=	percent deviation of the ith actual stiffness from its analytical value
$\delta\boldsymbol{k}'$	=	reduced stiffness correction vector
$[\delta M]$	=	mass correction matrix
$\delta\boldsymbol{m}$	=	mass correction vector
δm_i	=	percent deviation of the ith actual mass from its analytical value
$\delta\boldsymbol{m}'$	=	reduced mass correction vector
$[\delta X]$	=	$N \times n$ matrix whose ith column corresponds to $\delta\bar{\boldsymbol{x}}$ at ω_i
$\delta\boldsymbol{x}$	=	difference in response vectors between the actual system and the analytical model
$\delta\bar{\boldsymbol{x}}$	=	complex steady-state amplitude of $\delta\boldsymbol{x}$
ϵ_c	=	error parameter for the updated damping parameters
$(\epsilon_c)_0$	=	error parameter for the analytical damping parameters
ϵ_k	=	error parameter for the updated stiffnesses
$(\epsilon_k)_0$	=	error parameter for the analytical stiffnesses
ϵ_m	=	error parameter for the updated masses
$(\epsilon_m)_0$	=	error parameter for the analytical masses
$[\Omega]$	=	$n \times n$ diagonal matrix whose ith element corresponds to ω_i
ω_i	=	ith excitation frequency

Introduction

HIGHLY accurate and detailed finite element models are required to analyze and predict the dynamic behavior of complex structures during analysis and design. Once the finite element model of a physical system is constructed, its accuracy is often tested by comparing its analytical dynamic response predictions with the response obtained experimentally from the actual system. If the correlation between the two is poor, then assuming that the experimental measurements are correct, the analytical model must be adjusted so that the agreement between the analytical predictions and the test results is improved. The updated system may then be considered a better representation of the physical structure than the initial analytical model. The updated model can subsequently be used with more confidence to assess the stability and control characteristics and to predict the dynamic responses of the structure. The described process of correcting the system matrices is known as model updating.

In recent years, various methods have been developed to improve the quality of the analytical models using test data. A detailed discussion of every approach is beyond the scope of this paper, and interested readers are referred to survey papers by Imregun and Visser[1] and Mottershead and Friswell.[2] Whereas many different schemes have been used to update the analytical model of a physical structure, most did not explicitly enforce the connectivity information of the system matrices in their updating algorithms. Thus, fully

Received 11 June 2001; revision received 5 January 2002; accepted for publication 8 January 2002.

*Professor of Engineering, Department of Engineering. Member AIAA.
†Student, Department of Engineering.

populated system matrices may result from the updating procedure, and the updated model may or may not bear any resemblance to the physical system being analyzed. Assuming the analytical mass matrix to be accurate, Kabe[3] incorporated the structural connectivity information in addition to the free-response test data to adjust optimally the stiffness matrix. He utilized a Lagrange multipliers formalism to correct the stiffness matrix so that the percentage change to each stiffness element is minimized. Whereas Kabe's approach in updating the stiffness matrix is straightforward, his method is limited by the storage required and is highly computational intensive. In several recent papers,[4–6] the measured modes of vibration were used to adjust the analytical system mass and stiffness matrices. The conventional finite element optimal matrix storage scheme was employed to pass along the sparsity information, thereby enforcing the connectivity information, preserving the physical configuration of the structure, and reducing the size of the least-squares problems that need to be solved. More important, the proposed updating schemes all return adjusted system mass and stiffness parameters that are nearly identical to the exact values, provided the number of measured modes used in the update is sufficiently large.[4,6]

Most of the previous work on model updating used the experimental natural frequencies and mode shapes to correct the system parameters. Because only a limited number of modes of vibration can be measured, model updating based on free-response data results in underdetermined problems in most cases. Moreover, extracting the modes of vibration for structures with a high modal density can be very difficult. To overcome such deficiencies, researchers have turned to using the frequency-response data to correct the system matrices. Because frequency response can be gathered at any number of excitation frequencies, the numerical problems can be easily rendered overdetermined. In addition, the frequency-response data may be used directly to update the model without extracting the natural frequencies and mode shapes. Additionally, using frequency-response data to update the system matrices allows damping, which is present in all physical structures, to be accounted for in the analysis.

Utilizing frequency response data in addition to the connectivity information, Visser and Imregun[7] developed an iterative correction process to adjust the analytical model so that a unique solution that agreed with the measured frequency response was found. However, although the frequency response of the updated model matched that of the physical structure, the updated mass and stiffness parameters deviated substantially from those of the actual system in some cases. Cha and Tuck-Lee[8] also updated structural system parameters using frequency-response data. When the connectivity information was imposed, their updating algorithms returned updated results that tracked the actual system parameters accurately, despite the large deviations of the actual parameters from the analytical values. In their derivations, however, they assumed the damping matrix to be correct. In reality, the analytical and the actual damping matrices may be different.

In this paper, the structural connectivity information will be enforced to update the system mass, damping, and stiffness matrices in turn using frequency-response data. When the analytical and the actual system damping matrices are allowed to deviate from one another, the solution scheme becomes substantially more complicated mathematically and numerically. Matrix equations that govern the differences in the frequency response between the analytical and the actual systems are first solved. When the resulting equations are manipulated, the mass correction matrix can be isolated and solved using a least-squares solution scheme, which compensates for the number of measured frequency-response data being typically smaller than the degrees of freedom of the structure being analyzed. Enforcing the connectivity (or sparsity) information, the size of the least-squares problem can be drastically reduced, and the physical configuration of the system can also be preserved. Similar approaches are then used to isolate the damping and stiffness correction matrices. Because the correction matrices are isolated and solved in turn, errors in any correction matrix will not adversely affect the results of the other updates.

Proposed Model Updating Algorithm

In the subsequent derivations, it will be assumed that the frequency response of all of the coordinates can be measured. Because of physical limitations, however, the analytical output vector generally contains more points than are available from measurements. Thus, like other updating techniques, the incompleteness of the measured data gives rise to some difficulties. In this case, the analytical model must be reduced or the measured frequency-response data must be expanded so that the measured coordinates match those of the analytical model. Techniques for reducing the analytical model and for expanding the measured frequency-response data can be found in Refs. 9 and 10, respectively, and they will not be pursued here. In this paper, it will be assumed that all of the coordinates can be measured and that all of the data are correct. This allows one to focus attention on the quality of the proposed updating techniques and not confound the resulting updates with errors introduced by either model reduction or data expansion.

Consider a viscously damped system that is subjected to a forced harmonic excitation. The governing equation of its analytical model, with N degrees of freedom, is given by

$$[M_0]\ddot{\boldsymbol{x}}_0 + [C_0]\dot{\boldsymbol{x}}_0 + [K_0]\boldsymbol{x}_0 = \boldsymbol{f}e^{j\omega t} \tag{1}$$

where $[M_0]$, $[C_0]$, and $[K_0]$ are symmetric matrices of size $N \times N$ and both $\boldsymbol{x}_0$ and $\boldsymbol{f}$ are of length N. The matrix governing equation of the actual system, also with N degrees of freedom, under the same excitation is given by

$$[M]\ddot{\boldsymbol{x}} + [C]\dot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{f}e^{j\omega t} \tag{2}$$

where $[M]$, $[C]$, and $[K]$ are assumed to be symmetric. Because the forcing vector can be controlled and specified accurately, it is assumed that the analytical and the actual systems are subjected to the same external excitations.

When the measured frequency-response data are assumed to be correct, the actual and the analytical system matrices, along with their respective output vectors, may deviate from one another by some amount due to modeling errors, which may be caused by inappropriate modeling assumptions, uncertainties in material properties, and insufficient modeling details, such that

$$[M] = [M_0] + [\delta M], \qquad [C] = [C_0] + [\delta C]$$
$$[K] = [K_0] + [\delta K], \qquad \boldsymbol{x} = \boldsymbol{x}_0 + \delta\boldsymbol{x} \tag{3}$$

When Eq. (3) is substituted into Eq. (2), harmonic input and response are assumed, and Eq. (1) is noted, the following matrix equation in terms of the complex steady-state vibration amplitudes, $\bar{\boldsymbol{x}}$ and $\delta\bar{\boldsymbol{x}}$, is obtained:

$$\big([K_0] - \omega^2[M_0] + j\omega[C_0]\big)\delta\bar{\boldsymbol{x}} + ([\delta K] - \omega^2[\delta M] + j\omega[\delta C])\bar{\boldsymbol{x}} = \boldsymbol{0} \tag{4}$$

If n distinct frequency-response data points are measured, at excitation frequencies denoted by $(\omega_1, \omega_2, \ldots, \omega_n)$, then the set of n equations of Eq. (4) can be written in compact matrix form as

$$[P] + [\delta K][X] - [\delta M][X][\Omega^2] + j[\delta C][X][\Omega] = [0] \tag{5}$$

where $[P]$ is a known matrix of the form

$$[P] = [K_0][\delta X] - [M_0][\delta X][\Omega^2] + j[C_0][\delta X][\Omega] \tag{6}$$

When $[P]$, $[X]$, and $[\Omega]$ are known, the objective is to find $[\delta M]$, $[\delta K]$, and $[\delta C]$.

Because damping is present, Eq. (5) is complex. When only the real parts of Eq. (5) are considered, the following is obtained:

$$[P_{\mathcal{R}}] + [\delta K][X_{\mathcal{R}}] - [\delta M][X_{\mathcal{R}}][\Omega^2] - [\delta C][X_{\mathcal{I}}][\Omega] = [0] \tag{7}$$

where subscripts $\mathcal{R}$ and $\mathcal{I}$ denote the real and imaginary parts, respectively, and

$$[P_{\mathcal{R}}] = [K_0][\delta X_{\mathcal{R}}] - [M_0][\delta X_{\mathcal{R}}][\Omega^2] - [C_0][\delta X_{\mathcal{I}}][\Omega] \tag{8}$$

In general, none of the analytical system matrices will be exact. Thus, $[\delta M] \neq [0]$, $[\delta C] \neq [0]$, and $[\delta K] \neq [0]$, and an infinite number of $[\delta M]$, $[\delta C]$ and $[\delta K]$ combinations may satisfy Eq. (7). To isolate each correction matrix by itself, consider three sets of frequency-response data, each of which contains n distinct excitation frequencies and satisfies the following matrix equation:

$$[P_{\mathcal{R}i}]+[\delta K][X_{\mathcal{R}i}]-[\delta M][X_{\mathcal{R}i}]\left[\Omega_i^2\right]-[\delta C][X_{\mathcal{I}i}][\Omega_i]=[0] \quad (9)$$

for $i = 1, 2, 3$. All of the excitation frequencies in $[\Omega_i]$ are distinct from those in $[\Omega_j]$, for $i \neq j$, and all of the excitation frequencies lie within the frequency range of interest.

To isolate $[\delta M]$ and $[\delta C]$, first premultiply Eq. (9) for $i = 1$ by $[X_{\mathcal{R}2}]^T$, and then premultiply Eq. (9) for $i = 2$ by $[X_{\mathcal{R}1}]^T$. Then, take the transpose of the latter equation and subtract it from the former equation; a matrix equation that depends only on $[\delta M]$ and $[\delta C]$ is obtained:

$$\begin{aligned}&[X_{\mathcal{R}2}]^T[P_{\mathcal{R}1}]-[X_{\mathcal{R}2}]^T[\delta M][X_{\mathcal{R}1}]\left[\Omega_1^2\right]-[X_{\mathcal{R}2}]^T[\delta C][X_{\mathcal{I}1}][\Omega_1]\\&\quad-[P_{\mathcal{R}2}]^T[X_{\mathcal{R}1}]+\left[\Omega_2^2\right][X_{\mathcal{R}2}]^T[\delta M][X_{\mathcal{R}1}]\\&\quad+[\Omega_2][X_{\mathcal{I}2}]^T[\delta C][X_{\mathcal{R}1}]=[0]\end{aligned} \quad (10)$$

When a similar approach is used whereby Eq. (9) for $i = 1$ is premultiplied by $[X_{\mathcal{R}3}]^T$ and Eq. (9) for $i = 3$ is premultiplied by $[X_{\mathcal{R}1}]^T$, another matrix equation in terms of $[\delta M]$ and $[\delta C]$ is obtained:

$$\begin{aligned}&[X_{\mathcal{R}3}]^T[P_{\mathcal{R}1}]-[X_{\mathcal{R}3}]^T[\delta M][X_{\mathcal{R}1}]\left[\Omega_1^2\right]-[X_{\mathcal{R}3}]^T[\delta C][X_{\mathcal{I}1}][\Omega_1]\\&\quad-[P_{\mathcal{R}3}]^T[X_{\mathcal{R}1}]+\left[\Omega_3^2\right][X_{\mathcal{R}3}]^T[\delta M][X_{\mathcal{R}1}]\\&\quad+[\Omega_3][X_{\mathcal{I}3}]^T[\delta C][X_{\mathcal{R}1}]=[0]\end{aligned} \quad (11)$$

Equations (10) and (11) are then expanded so that $[\delta M]$ and $[\delta C]$ appear in vector forms:

$$[A_{11}]\delta\boldsymbol{c}+[A_{12}]\delta\boldsymbol{m}=\boldsymbol{b}_1 \quad (12)$$

$$[A_{21}]\delta\boldsymbol{c}+[A_{22}]\delta\boldsymbol{m}=\boldsymbol{b}_2 \quad (13)$$

Submatrices $[A_{ij}]$ are of size $n^2 \times N^2$, vectors $\boldsymbol{b}_i$ are of length n^2, and

$$\delta\boldsymbol{m}=[\delta m_{11}\cdots\delta m_{1N}\mid\delta m_{21}\cdots\delta m_{2N}\mid\cdots\mid\delta m_{N1}\cdots\delta m_{NN}]^T \quad (14)$$

where δm_{ij} corresponds to the (i, j)th element of $[\delta M]$. A similar expression exists for $\delta\boldsymbol{c}$.

At this stage, the optimal matrix storage scheme commonly used in finite elements[11] is utilized to pass along the well known and readily available sparsity information of the analytical system. Enforcing the structural connectivity information preserves the physical configuration of the system and imposes the condition that all of the zero elements in the analytical system matrices remain zeros in the adjusted system matrices. Mathematically, this is achieved by eliminating all of the known zero elements from $\delta\boldsymbol{m}$ and $\delta\boldsymbol{c}$ and by deleting all the corresponding columns in the $[A_{ij}]$. This drastically reduces the size of the problem to be solved. More important, when the connectivity information is enforced, the quality of the updates is significantly improved because only the appropriate nonzero elements are adjusted.[8]

Let $\delta\boldsymbol{m}'$, $\delta\boldsymbol{c}'$, and $[A'_{ij}]$ be the reduced correction vectors and reduced submatrices obtained by enforcing the connectivity information of the system. Then Eqs. (12) and (13) reduce to

$$[A'_{11}]\delta\boldsymbol{c}'+[A'_{12}]\delta\boldsymbol{m}'=\boldsymbol{b}_1 \quad (15)$$

$$[A'_{21}]\delta\boldsymbol{c}'+[A'_{22}]\delta\boldsymbol{m}'=\boldsymbol{b}_2 \quad (16)$$

From Eq. (16), $\delta\boldsymbol{c}'$ is given by

$$\delta\boldsymbol{c}'=[A'_{21}]^\dagger(\boldsymbol{b}_2-[A'_{22}]\delta\boldsymbol{m}') \quad (17)$$

where $[A'_{21}]^\dagger$ is the pseudoinverse of $[A'_{21}]$. (See the Appendix for a formal definition of the pseudoinverse of any matrix.) Substituting Eq. (17) into Eq. (15) leads to the following equation:

$$\left([A'_{12}]-[A'_{11}][A'_{21}]^\dagger[A'_{22}]\right)\delta\boldsymbol{m}'=\boldsymbol{b}_1-[A'_{11}][A'_{21}]^\dagger\boldsymbol{b}_2 \quad (18)$$

which can be expressed alternatively in the following compact form:

$$[A'_m]\delta\boldsymbol{m}'=\boldsymbol{r}_m \quad (19)$$

The entries of $[A'_m]$ and $\boldsymbol{r}_m$ are determined by expanding the left- and right-hand sides of Eq. (18), and the least-squares solution of Eq. (19) yields the mass corrections.

It appears that Eq. (17) may be used to solve for the damping corrections directly. However, because $\delta\boldsymbol{c}'$ depends on $\delta\boldsymbol{m}'$ explicitly, any errors in $\delta\boldsymbol{m}'$ will adversely affect the integrity of $\delta\boldsymbol{c}'$. Thus, to update the damping correction vector $\delta\boldsymbol{c}'$, another approach must be taken. Consider again Eqs. (15) and (16). From Eq. (16), $\delta\boldsymbol{m}'$ is given by

$$\delta\boldsymbol{m}'=[A'_{22}]^\dagger(\boldsymbol{b}_2-[A'_{21}]\delta\boldsymbol{c}') \quad (20)$$

Substituting Eq. (20) into Eq. (15) results in the following equation:

$$\left([A'_{11}]-[A'_{12}][A'_{22}]^\dagger[A'_{21}]\right)\delta\boldsymbol{c}'=\boldsymbol{b}_1-[A'_{12}][A'_{22}]^\dagger\boldsymbol{b}_2 \quad (21)$$

which can also be expressed in the following concise form:

$$[A'_c]\delta\boldsymbol{c}'=\boldsymbol{r}_c \quad (22)$$

Note that Eq. (21) depends only on the reduced submatrices $[A'_{ij}]$ and is independent of $\delta\boldsymbol{m}'$. Solving the least-squares problem of Eq. (22) readily yields the damping corrections.

In theory, Eqs. (15) and (16) can be expressed in matrix form $[A']\boldsymbol{y}=\boldsymbol{b}$ and solved simultaneously for $\boldsymbol{y}$, which contains the unknowns $\delta\boldsymbol{c}'$ and $\delta\boldsymbol{m}'$, as the least-squares solution. In this case, a solution vector $\boldsymbol{y}$ is sought such that the Euclidean norm of the residual vector $\|[A']\boldsymbol{y}-\boldsymbol{b}\|$ is minimized. If the least-squares problem has more than one solution, the one having the minimum Euclidean norm, that is, the minimum-norm solution, is sought. Clearly, the minimum-norm solution $\boldsymbol{y}$ will be different from the minimum-norm solutions $\delta\boldsymbol{c}'$ and $\delta\boldsymbol{m}'$ found separately by solving Eqs. (19) and (22), respectively. Numerical experiments showed that more accurate solution is obtained by solving two separate least-squares problems [see Eqs. (19) and (22)] as opposed to solving one large least-squares problem obtained by combining Eqs. (15) and (16).

To update the stiffness corrections $\delta\boldsymbol{k}'$, first premultiply and postmultiply Eq. (9) for $i = 1$ by $[X_{\mathcal{I}2}]^T$ and $[\Omega_1]^{-1}$, then premultiply and postmultiply Eq. (9) for $i = 2$ by $[X_{\mathcal{I}1}]^T$ and $[\Omega_2]^{-1}$, such that the following equations are obtained:

$$\begin{aligned}&[X_{\mathcal{I}2}]^T[P_{\mathcal{R}1}][\Omega_1]^{-1}+[X_{\mathcal{I}2}]^T[\delta K][X_{\mathcal{R}1}][\Omega_1]^{-1}\\&\quad-[X_{\mathcal{I}2}]^T[\delta M][X_{\mathcal{R}1}][\Omega_1]=[X_{\mathcal{I}2}]^T[\delta C][X_{\mathcal{I}1}]\end{aligned} \quad (23)$$

$$\begin{aligned}&[X_{\mathcal{I}1}]^T[P_{\mathcal{R}2}][\Omega_2]^{-1}+[X_{\mathcal{I}1}]^T[\delta K][X_{\mathcal{R}2}][\Omega_2]^{-1}\\&\quad-[X_{\mathcal{I}1}]^T[\delta M][X_{\mathcal{R}2}][\Omega_2]=[X_{\mathcal{I}1}]^T[\delta C][X_{\mathcal{I}2}]\end{aligned} \quad (24)$$

Taking the transpose of Eq. (24) and equating the left-hand sides of the resultant and Eq. (23) yields

$$\begin{aligned}&[X_{\mathcal{I}2}]^T[P_{\mathcal{R}1}][\Omega_1]^{-1}+[X_{\mathcal{I}2}]^T[\delta K][X_{\mathcal{R}1}][\Omega_1]^{-1}\\&\quad-[X_{\mathcal{I}2}]^T[\delta M][X_{\mathcal{R}1}][\Omega_1]=[\Omega_2]^{-1}[P_{\mathcal{R}2}]^T[X_{\mathcal{I}1}]\\&\quad+[\Omega_2]^{-1}[X_{\mathcal{R}2}]^T[\delta K][X_{\mathcal{I}1}]-[\Omega_2][X_{\mathcal{R}2}]^T[\delta M][X_{\mathcal{I}1}]\end{aligned} \quad (25)$$

Note that Eq. (25) consists of only two correction matrices, $[\delta M]$ and $[\delta K]$. When a similar approach is used whereby Eq. (9) for $i = 1$ is premultiplied by $[X_{\mathcal{I}3}]^T$ and postmultiplied by $[\Omega_1]^{-1}$ and Eq. (9) for $i = 3$ is premultiplied by $[X_{\mathcal{I}1}]^T$ and postmultiplied by $[\Omega_3]^{-1}$, another matrix equation in terms of $[\delta M]$ and $[\delta K]$ is obtained:

$$\begin{aligned}&[X_{\mathcal{I}3}]^T[P_{\mathcal{R}1}][\Omega_1]^{-1}+[X_{\mathcal{I}3}]^T[\delta K][X_{\mathcal{R}1}][\Omega_1]^{-1}\\&\quad-[X_{\mathcal{I}3}]^T[\delta M][X_{\mathcal{R}1}][\Omega_1]=[\Omega_3]^{-1}[P_{\mathcal{R}3}]^T[X_{\mathcal{I}1}]\\&\quad+[\Omega_3]^{-1}[X_{\mathcal{R}3}]^T[\delta K][X_{\mathcal{I}1}]-[\Omega_3][X_{\mathcal{R}3}]^T[\delta M][X_{\mathcal{I}1}]\end{aligned} \quad (26)$$

When Eqs. (25) and (26) are expanded so that $[\delta M]$ and $[\delta K]$ appear in vector forms and the connectivity information is imposed, the following equations are obtained:

$$[E'_{11}]\delta \boldsymbol{k}' + [E'_{12}]\delta \boldsymbol{m}' = \boldsymbol{q}_1 \tag{27}$$

$$[E'_{21}]\delta \boldsymbol{k}' + [E'_{22}]\delta \boldsymbol{m}' = \boldsymbol{q}_2 \tag{28}$$

Solving for $\delta\boldsymbol{m}'$ in terms of $\delta\boldsymbol{k}'$ using Eq. (28) and substituting the resulting expression of $\delta\boldsymbol{m}'$ into Eq. (27) yield a problem of the form

$$[A'_k]\delta \boldsymbol{k}' = \boldsymbol{r}_k \tag{29}$$

whose least-squares solution corresponds to the stiffness corrections.

Because the reduced correction vectors, $\delta\boldsymbol{m}'$, $\delta\boldsymbol{c}'$, and $\delta\boldsymbol{k}'$, are obtained numerically, the corresponding correction matrices, $[\delta M]$, $[\delta C]$, and $[\delta K]$, are generally nonsymmetric. To enforce the symmetry condition, the expressions in Eq. (3) are slightly modified as follows:

$$[M_{\text{update}}] = [M_0] + \tfrac{1}{2}([\delta M] + [\delta M]^T)$$

$$[C_{\text{update}}] = [C_0] + \tfrac{1}{2}([\delta C] + [\delta C]^T)$$

$$[K_{\text{update}}] = [K_0] + \tfrac{1}{2}([\delta K] + [\delta K]^T) \tag{30}$$

To illustrate how the connectivity information is imposed, consider a system whose mass matrix is diagonal. Then $\delta m_{ij} = 0$ for $i \neq j$, and $\delta\boldsymbol{m}$ reduces to

$$\delta \boldsymbol{m}' = [\delta m_{11} \quad \delta m_{22} \quad \cdots \quad \delta m_{NN}]^T \tag{31}$$

in which case the reduced submatrices $[A'_{12}]$ and $[A'_{22}]$ are obtained from $[A_{12}]$ and $[A_{22}]$ by deleting all of the columns that multiply by δm_{ij} for $i \neq j$. Thus, the initial $n^2 \times N^2$ submatrices $[A_{12}]$ and $[A_{22}]$ are reduced to ones of size $n^2 \times N$. Similarly, if the damping matrix is tridiagonal, then $\delta c_{ij} = 0$ for $|i - j| > 1$, and

$$\delta \boldsymbol{c}' = [\delta c_{11} \quad \delta c_{12} | \delta c_{21} \quad \delta c_{22} \quad \delta c_{23} | \cdots | \delta c_{NN-1} \quad \delta c_{NN}]^T \tag{32}$$

in which case $[A'_{11}]$ and $[A'_{21}]$ are obtained from $[A_{11}]$ and $[A_{21}]$ by deleting all of the columns that multiply by δc_{ij} for $|i - j| > 1$. Thus, the initial $n^2 \times N^2$ submatrices $[A_{11}]$ and $[A_{21}]$ are reduced to ones of $n^2 \times (3N - 2)$. A similar approach is used to impose the structural connectivity information of the stiffness matrix.

A few words regarding the structural connectivity information are warranted here. The proposed updating algorithms depend on the connectivities in the analytical matrices to be correct. Because the basis of model updating is the analytical model, the analytical model must capture certain physical attributes of the actual system. Here, it is assumed that the analytical model and the actual system share the same sparsity pattern. The analytical model for a physical structure is by no means unique. Given a physical structure, different assumptions or different degrees of idealization can result in different analytical models and, hence, different connectivity information. Not surprisingly, different analytical models will result in different corrections. Nevertheless, for a given analytical model, the proposed updating algorithms will return an adjusted model that is substantially more accurate than the initial analytical one.

Finally, even if the system matrices are full (in which case there are no elements in $\delta\boldsymbol{m}$, $\delta\boldsymbol{c}$, and $\delta\boldsymbol{k}$ to delete), the proposed schemes are still valid and useful. When the number of excitation frequencies is not equal to the number of degrees of freedom of the analytical model, the coefficient matrices $[X_{\mathcal{R}i}]$ and $[X_{\mathcal{I}i}]$ of Eqs. (10), (11), (25), and (26) are rectangular and cannot be inverted. Thus, it may be difficult to find expressions for $[\delta M]$, $[\delta C]$, and $[\delta K]$ explicitly. In this case, however, the proposed algorithms lead to three least-squares problems similar to those of Eqs. (19), (22), and (29), which can be readily solved to update the mass, damping, and stiffness matrices in turn.

Numerical Issues

The numerical issues encountered when solving a least-squares problem depend on whether the problem is underdetermined or overdetermined. A full rank, underdetermined, least-squares problem has an infinite number of solutions, whereas a full rank, overdetermined, least-squares problem has a solution that is unique.[12] Because the reduced submatrices $[A'_{ij}]$ were obtained by deleting certain columns of $[A_{ij}]$ to enforce the sparsity information, $[A'_m]$ of Eq. (19) may suffer rank deficiency. Fortunately, numerical techniques exist for determining the numerical rank of such systems and subsequently finding the unique minimum norm solutions. Thus, regardless if the system is overdetermined or underdetermined, has full rank or is rank deficient, the system mass, damping, and stiffness matrices can always be updated numerically.

Numerical experiments revealed that the least-squares solution of an overdetermined system is more accurate than an underdetermined system, even if the system is rank deficient.[8] The previous observation has direct implications about the minimum n that should be obtained for each data set before implementing the proposed model updating algorithm. For the system of Fig. 1, $[A'_m]$ of Eq. (19) is of size $n^2 \times N$, and $[A'_c]$ of Eq. (22) and $[A'_k]$ of Eq. (29) are both of size $n^2 \times (3N - 2)$. Whereas it appears that $n \geq \sqrt{N}$ is needed to updated the mass matrix and $n \geq \sqrt{(3N - 2)}$ is required to updated the damping and stiffness matrices, note that because $\delta\boldsymbol{m}'$ and $\delta\boldsymbol{c}'$ are coupled [see Eqs. (15) and (16)], the larger of the two n dictates the minimum number of data points that are needed to perform the update. Thus, to update the mass matrix, the number of frequency-response data points n must also be greater or equal to $\sqrt{(3N - 2)}$. Based on the earlier discussion, for the system of Fig. 1, at least $n \geq \sqrt{(3N - 2)}$ data points in each data set are required to update the system parameters accurately.

Results

To update the system mass, damping, and stiffness parameters requires the solution of three least-squares problems [see Eqs. (19), (22), and (29)]. For the system of Fig. 1 (whose mass matrix is diagonal and whose damping and stiffness matrices are tridiagonal), $[A'_m]$ of Eq. (19) is of size $n^2 \times N$, and vectors $\delta\boldsymbol{m}'$ and $\boldsymbol{r}_m$ are of lengths N and n^2, respectively. Matrices $[A'_c]$ of Eq. (22) and $[A'_k]$ of Eq. (29) are both of size $n^2 \times (3N - 2)$, vectors $\delta\boldsymbol{c}'$ and $\delta\boldsymbol{k}'$ are of length $3N - 2$, and $\boldsymbol{r}_c$ and $\boldsymbol{r}_k$ are of length n^2.

When solving least-squares problems, the MATLAB® routine pinv was implemented, which computes the Moore–Penrose pseudoinverse (see Ref. 13) of a matrix and can be used regardless if the coefficient matrix of the least-squares problem has full rank or is rank deficient. If the least-squares problem has an infinite number of solutions, pinv returns a solution vector that has minimum norm, which can be used as the appropriate corrections to the structural parameters. The proposed method is applied to a structure whose mass matrix is diagonal. Although restrictive at first glance, note that

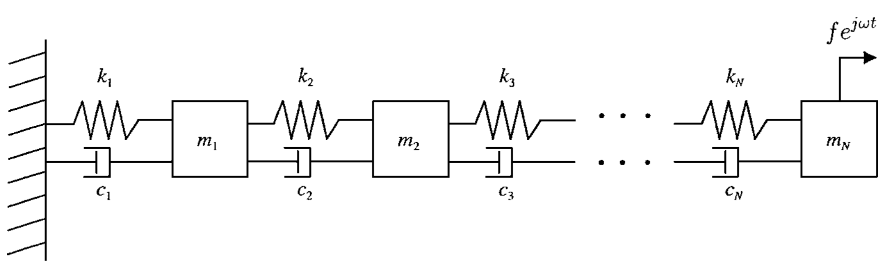

Fig. 1 Simple chain of coupled oscillators.

lumped or diagonal mass matrices are often used in practice because of their general economy and because the numerical operations for the solution of the dynamic equations are generally significantly reduced. A discussion on how to reduce consistent mass matrices into lumped mass matrices may be found in Ref. 14.

The analytical and the actual mass, stiffness, and damping parameters for the system of Fig. 1 are given by (m_0, k_0, c_0) and (m_i, k_i, c_i), respectively. A single harmonic force is applied to the right end of the structure, and the harmonic excitation is assumed to be identical for both the analytical and the actual systems. Thus, $\boldsymbol{f} = [0\ \ 0\ \cdots\ f]^T$, where f is the magnitude of the harmonic excitation. Because the excitation location remains fixed and only the excitation frequency is varied, the proposed updating algorithm can be easily implemented experimentally.

The proposed updating algorithms are applied to the system of Fig. 1 for $N = 26$, whose analytical parameters are $m_0 = 2.00$ kg, $c_0 = 0.50$ N·s/m, and $k_0 = 10.00$ N/m. The actual and the analytical mass, damping, and stiffness parameters are related by $m_i = m_0(1 + \delta m_i)$, $c_i = c_0(1 + \delta c_i)$, and $k_i = k_0(1 + \delta k_i)$. The actual mass, damping, and stiffness parameters are listed in Tables 1–3, respectively. In general, the $3n$ excitation frequencies needed to execute the updating algorithms are chosen such that they lie within the frequency range of interest. For the system under consideration, the desired frequencies are equally spaced between the interval of 0 and $2\sqrt{(k_0/m_0)}$, which corresponds to the lowest and the highest natural frequencies for the infinitely long analytical model of Fig. 1.[15]

Table 1 Actual and updated masses for $n = N = 26$[a]

m_{actual}, kg	m_{update}, kg	m_{actual}, kg	m_{update}, kg
$m_1 = 2.1831$	2.1831	$m_{14} = 1.3355$	1.3355
$m_2 = 1.3117$	1.3117	$m_{15} = 2.6680$	2.6680
$m_3 = 2.6581$	2.6581	$m_{16} = 1.3899$	1.3899
$m_4 = 2.4371$	2.4371	$m_{17} = 1.8632$	1.8632
$m_5 = 1.7651$	1.7651	$m_{18} = 1.6231$	1.6231
$m_6 = 2.8502$	2.8502	$m_{19} = 1.1578$	1.1578
$m_7 = 1.7984$	1.7984	$m_{20} = 1.1439$	1.1439
$m_8 = 1.7793$	1.7793	$m_{21} = 1.8189$	1.8189
$m_9 = 2.7588$	2.7588	$m_{22} = 1.2320$	1.2320
$m_{10} = 2.1221$	2.1221	$m_{23} = 1.6389$	1.6389
$m_{11} = 1.1613$	1.1613	$m_{24} = 2.2254$	2.2254
$m_{12} = 2.0234$	2.0234	$m_{25} = 2.3068$	2.3068
$m_{13} = 2.6722$	2.6722	$m_{26} = 1.9261$	1.9261

[a]Analytical mass is $m_0 = 2.00$ kg.

Table 2 Actual and updated damping parameters for $n = N = 26$[a]

c_{actual}, N·s/m	c_{update}, N·s/m	c_{actual}, N·s/m	c_{update}, N·s/m
$c_1 = 0.4940$	0.4940	$c_{14} = 0.3576$	0.3576
$c_2 = 0.4029$	0.4929	$c_{15} = 0.5853$	0.5853
$c_3 = 0.3340$	0.3340	$c_{16} = 0.6998$	0.6998
$c_4 = 0.6061$	0.6061	$c_{17} = 0.4027$	0.4027
$c_5 = 0.4311$	0.4311	$c_{18} = 0.3497$	0.3497
$c_6 = 0.5435$	0.5435	$c_{19} = 0.6712$	0.6712
$c_7 = 0.3389$	0.3389	$c_{20} = 0.5348$	0.5348
$c_8 = 0.5847$	0.5847	$c_{21} = 0.3772$	0.3772
$c_9 = 0.4838$	0.4838	$c_{22} = 0.6881$	0.6881
$c_{10} = 0.4252$	0.4252	$c_{23} = 0.5227$	0.5227
$c_{11} = 0.5410$	0.5410	$c_{24} = 0.6158$	0.6158
$c_{12} = 0.6007$	0.6007	$c_{25} = 0.6666$	0.6666
$c_{13} = 0.5602$	0.5602	$c_{26} = 0.6099$	0.6099

[a]Analytical damping parameter is $c_0 = 0.50$ N·s/m.

Table 3 Actual and updated stiffnesses for $n = N = 26$[a]

k_{actual}, N/m	k_{update}, N/m	k_{actual}, N/m	k_{update}, N/m
$k_1 = 8.2800$	8.2800	$k_{14} = 11.9754$	11.9754
$k_2 = 13.7603$	13.7603	$k_{15} = 10.9946$	10.9946
$k_3 = 11.2103$	11.2103	$k_{16} = 11.8966$	11.8966
$k_4 = 13.0215$	13.0215	$k_{17} = 10.0639$	10.0639
$k_5 = 5.8686$	5.8686	$k_{18} = 12.7216$	12.7216
$k_6 = 14.2652$	14.2652	$k_{19} = 12.5452$	12.5452
$k_7 = 6.6143$	6.6143	$k_{20} = 14.1144$	14.1144
$k_8 = 6.5972$	6.5972	$k_{21} = 13.2052$	13.2052
$k_9 = 12.4041$	12.4041	$k_{22} = 9.8652$	9.8652
$k_{10} = 13.2798$	13.2798	$k_{23} = 12.7865$	12.7865
$k_{11} = 11.8978$	11.8978	$k_{24} = 11.6008$	11.6008
$k_{12} = 12.6406$	12.6406	$k_{25} = 13.1870$	13.1870
$k_{13} = 6.6714$	6.6714	$k_{26} = 6.4709$	6.4709

[a]Analytical stiffness is $k_0 = 10.00$ N/m.

Tables 1–3 compare the actual and the updated structural parameters, obtained by imposing the structural connectivity information, for $n = 26$. For the purpose of numerical simulations, the δm_i, δc_i, and δk_i are given by three sets of uniformly distributed random variables, with a mean and a standard deviation of (−0.0413, 0.2708), (0.0329, 0.2321), and (0.1075, 0.2659), respectively. From the results of Table 1, note the exact agreement between the updated values and the actual masses, despite the large deviations of the actual masses from the analytical values. Because the sparsity information is imposed during the update, all of the zero terms in the initial mass matrix remain zeros, thus preserving the physical configuration of the structure. Tables 2 and 3 show the actual and the updated damping and stiffness parameters for the system of Fig. 1. When the tridiagonality condition is imposed, the updated damping and stiffness values track the actual damping and stiffness parameters exactly, even though the deviations between the actual and the analytical damping coefficients and stiffnesses are large.

Incidentally, note that, when $n = N$ and if the connectivity information is not enforced, the updated system matrices will be different from the exact matrices. This was shown by Cha and Tuck-Lee[8] for the case where the damping matrix is exact. Numerical experiments have shown that failing to impose the connectivity information has similar results in this case, where damping is also being updated. In addition, when the connectivity information is not imposed, the correction matrices will be fully populated, which destroys the physical configuration of the structure.

When the number of measured frequency-response data points in each data set is equal to the size of the analytical model, that is, when $n = N$, the updating algorithms can be used to correct accurately the analytical mass, damping, and stiffness parameters of the system, provided that the connectivity information is imposed (Tables 1–3). Because of physical limitations, time, or cost constraints, however, the number of distinct frequencies is almost always less than the degrees of freedom of the analytical model. Thus, in application, $n < N$. Of considerable interest, then, is how the number of data points n affect the quality of the updated systems parameters.

To quantify the accuracy of the mass updating algorithm, the following relative error parameters for the updated and the analytical masses are introduced:

$$\epsilon_m = \frac{|\boldsymbol{m}_{\text{update}} - \boldsymbol{m}_{\text{actual}}|}{|\boldsymbol{m}_{\text{actual}}|}, \qquad (\epsilon_m)_0 = \frac{|\boldsymbol{m}_{\text{analytical}} - \boldsymbol{m}_{\text{actual}}|}{|\boldsymbol{m}_{\text{actual}}|} \tag{33}$$

where $\boldsymbol{m}_{\text{update}}$, $\boldsymbol{m}_{\text{analytical}}$, and $\boldsymbol{m}_{\text{actual}}$ are vectors of length N whose elements are the updated, analytical, and actual lumped masses, respectively, and $|\boldsymbol{a}|$ is the Euclidean norm of the vector $\boldsymbol{a}$. Similar expressions are also defined for the relative error parameters for the damping and stiffness parameters. For an updated model to be judged better than the initial analytical model, $\epsilon_m < (\epsilon_m)_0$, $\epsilon_c < (\epsilon_c)_0$, and $\epsilon_k < (\epsilon_k)_0$. Additionally, the smaller the error parameters are, the better the update.

Figure 2 shows the variations of ϵ_m, ϵ_c, and ϵ_k, for the system parameters of Tables 1–3, as a function of the number of frequency-response data points, n. Also shown are the corresponding $(\epsilon_m)_0$, $(\epsilon_c)_0$, and $(\epsilon_k)_0$, which are independent of n and are given by the horizontal lines. In general, the updated parameters become more accurate as n increases. The experimental results are consistent with physical intuition: The more information that can be gathered about the frequency response of the physical system, the better the updated model becomes.

The curves of Fig. 2 also reveal the minimum number of distinct excitation frequencies in each data set that are needed to achieve a certain level of accuracy. Thus, it can be used to determine the fewest number of frequency responses in each data set that should be obtained for the purpose of performing the update. From numerical experiments, it was observed that more accurate solutions are

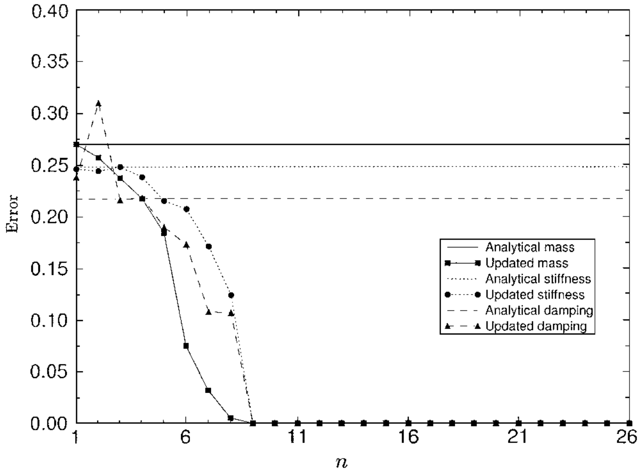

Fig. 2 Mass, damping, and stiffness error parameters, ϵ_m, ϵ_c, and ϵ_k, as a function of the number of frequency-response data points n for Tables 1–3 system parameters: horizontal lines represent mass, damping, and stiffness error parameters of the analytical model, $(\epsilon_m)_0$, $(\epsilon_c)_0$, and $(\epsilon_k)_0$.

obtained when the least-squares system becomes overdetermined. Thus, for $N = 26$, the discussion in the preceding section predicts that at least nine data points (for each data set) are needed to update the system parameters to ensure sufficient accuracy. Note that the results of Fig. 2 validate the heuristic criteria regarding the fewest frequency-response data points that need to be measured. Indeed, for $n \geq \sqrt{(3N-2)}$ or $n \geq 9$, the relative error parameters become negligible, which implies that the updates are nearly identical to the actual structural parameters. The numerical results also clearly indicate that there is a saturation point beyond which additional information does not lead to any significant improvement in the corrected matrices.

To determine the corrections, three least-squares problems of the form $[A]\boldsymbol{x} = \boldsymbol{r}$ need to be solved. For the system of Fig. 1 and for $n < \sqrt{(3N-2)}$, the least-squares problems become underdetermined and may also suffer rank deficiency. For such rank deficient systems, small changes in $[A]$ and $\boldsymbol{r}$ can induce arbitrarily large changes in the least-squares solution of $\boldsymbol{x}$ (Ref. 12). Thus, when $n < \sqrt{(3N-2)}$, the updated structural parameters may deviate substantially from the actual values and may be worse than the initial analytical values. Figure 2 clearly illustrates these phenomena.

Figure 3 shows the normalized amplitude of the frequency response of the first mass, $|x_1/f|$, for the analytical, actual, and

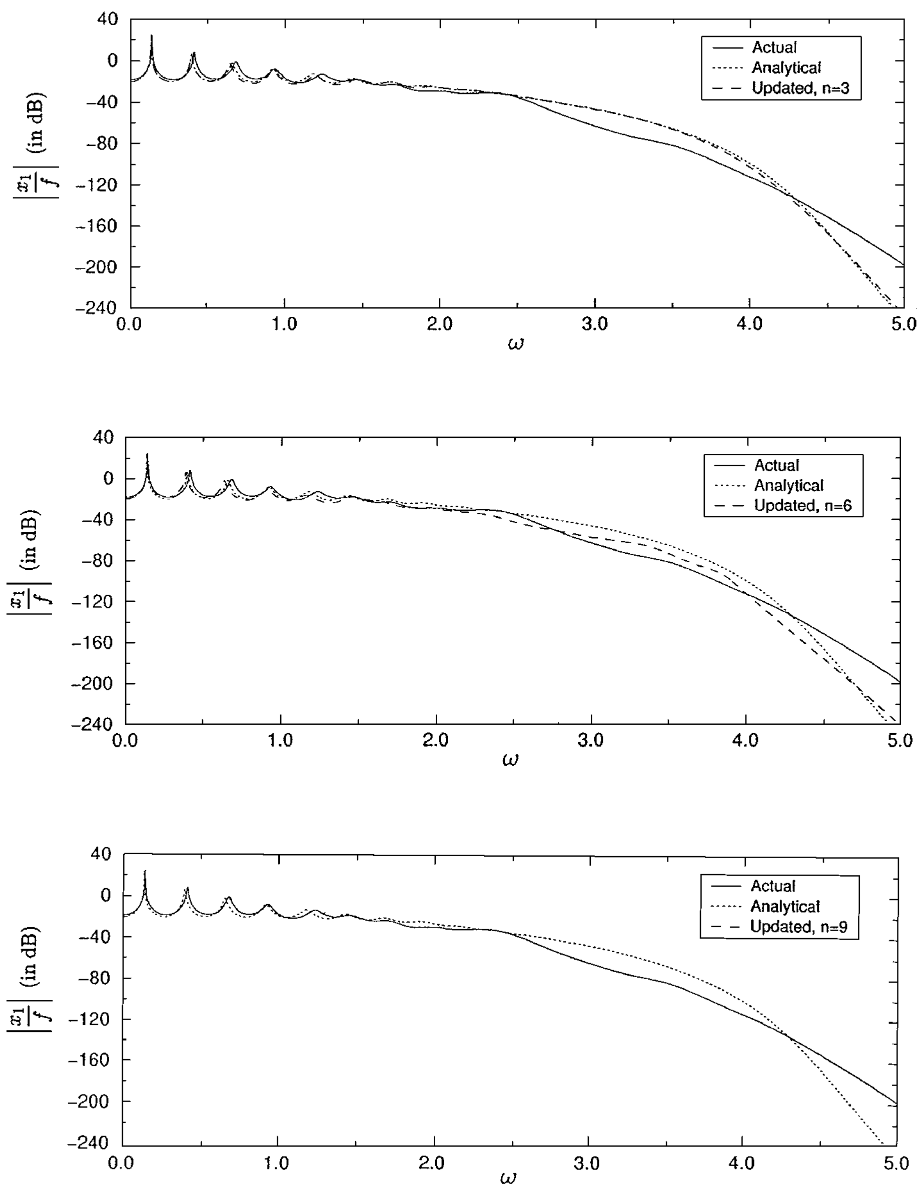

Fig. 3 Normalized amplitude of the frequency response of the first mass $|x_1/f|$ (in decibels) for the analytical, actual, and updated systems of Fig. 1, for $n = 3, 6$, and 9. System parameters are identical to those of Tables 1–3.

updated systems of Fig. 1, for $n = 3$, 6, and 9. The system parameters are identical to those of Tables 1–3. Because the deviations between the actual and the analytical system parameters are large, the frequency response of the actual system deviates substantially from the analytical model. For $n = 3$, the frequency response of the updated model is merely a perturbation of that of the analytical system, and it deviates considerably from the actual system, especially at high excitation frequencies. For $n = 6$, the frequency response of the updated model still differs from the physical system, but it improves on the frequency response of the updated structure obtained with $n = 3$ over certain frequency ranges. For $n = 9$, the updated model reproduces the frequency response of the actual system over the entire range of excitation frequencies considered. Thus, as n (the number of frequency-response data points in each data set used to update the structural parameters of the system) increases, the frequency response of the corrected system becomes nearly identical to that of the actual structure.

Conclusions

New mass, damping, and stiffness updating algorithms are developed to correct the structural system parameters using frequency-response data. When the differences between the measured and the analytical frequency-response data are utilized as inputs, the structural system parameters of the analytical model can be accurately updated or corrected, provided that the number of measured frequency-response data points is sufficiently large. When the unknown mass, damping, and stiffness correction matrices as are expressed column vectors, the structural connectivity information are easily imposed, which preserves the physical configuration of the system and reduces the amount of computational effort required to correct the analytical model. Moreover, the proposed updating schemes result in least-squares problems whose structures reveal the minimum number of frequency-response data points that need to be measured to ensure a sufficiently accurate updated model. Numerical experiments show the accuracy of the updating algorithms and verify the utility of the proposed approaches of correcting the structural system parameters using frequency-response data.

Appendix: Pseudoinverse

A formal definition for the pseudoinverse of any matrix $[A]$, regardless if it has full rank or is rank deficient, can be given as follows[12]: Let

$$[A] = [U][\Sigma][V]^T$$

be the singular value decomposition of $[A]$. Then its pseudoinverse $[A]^\dagger$, which always exists and is unique, is given by[12]

$$[A]^\dagger = [V][\Sigma]^\dagger[U]^T$$

where

$$[\Sigma]^\dagger = \mathrm{diag}(1/\sigma_1, 1/\sigma_2, \ldots, 1/\sigma_r, 0, \ldots, 0)$$

where $\sigma_1, \ldots, \sigma_r$ correspond to the r nonzero singular values of $[A]$.

References

[1] Imregun, M., and Visser, W. J., "A Review of Model Updating Techniques," *Shock and Vibration Digest*, Vol. 23, No. 1, 1991, pp. 9–20.

[2] Mottershead, J. E., and Friswell, M. I., "Model Updating in Structural Dynamics: A Survey," *Journal of Sound and Vibration*, Vol. 167, No. 2, 1993, pp. 347–375.

[3] Kabe, A. M., "Stiffness Matrix Adjustment Using Mode Data," *AIAA Journal*, Vol. 23, No. 9, 1985, pp. 1431–1436.

[4] Cha, P. D., "Correcting System Matrices Using the Orthogonality Conditions of Distinct Measured Modes," *AIAA Journal*, Vol. 38, No. 4, 2000, pp. 730–732.

[5] Cha, P. D., and Gu, W., "Model Updating Using an Incomplete Set of Measured Modes," *Journal of Sound and Vibration*, Vol. 233, No. 4, 2000, pp. 587–600.

[6] Cha, P. D., and dePillis, L. G., "Model Updating by Adding Known Masses," *International Journal for Numerical Methods in Engineering*, Vol. 50, No. 11, 2001, pp. 2547–2571.

[7] Visser, W. J., and Imregun, M., "A Technique to Update Finite Element Models Using Frequency Response Data," *Proceedings of the 9th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 1991, pp. 462–468.

[8] Cha, P. D., and Tuck-Lee, J. P., "Updating Structural System Parameters Using Frequency Response Data," *Journal of Engineering Mechanics*, Vol. 126, No. 12, 2000, pp. 1240–1246.

[9] Goh, E. L., and Mottershead, J. E., "On Model Reduction Techniques for Finite Element Updating," *NAFEMS/DTA International Conference on Structural Dynamics, Modelling, Test, Analysis, and Correlation*, NAFEMS, Glasgow, Scotland, U.K., 1993, pp. 421–432.

[10] Tranxuan, D., He, J., and Choudhury, R., "Damage Detection of Truss Structures Using Measured Frequency Response Function Data," *Proceedings of the 15th International Modal Analysis Conference*, Society for Experimental Mechanics, Bethel, CT, 1997, pp. 961–967.

[11] Bathe, K. J., *Finite Element Procedures in Engineering Analysis*, Prentice–Hall, Upper Saddle River, NJ, 1996, pp. 19–21.

[12] Golub, G. H., and Van Loan, C. F., *Matrix Computations*, 2nd ed., Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 193–259.

[13] Rao, C. R., and Mitra, S. K., *Generalized Inverse of Matrices and Its Applications*, Wiley, New York, 1971.

[14] Hughes, T. J. R., *The Finite Element Method*, Prentice–Hall, Englewood Cliffs, NJ, 1987, pp. 436–446.

[15] Chen, F. Y., "On Modeling and Direct Solution of Certain Free Vibration Systems," *Journal of Sound and Vibration*, Vol. 14, No. 1, 1971, pp. 57–79.

A. Berman
Associate Editor